# USING INTEGRATION TO FIND AREAS UNDER CURVES

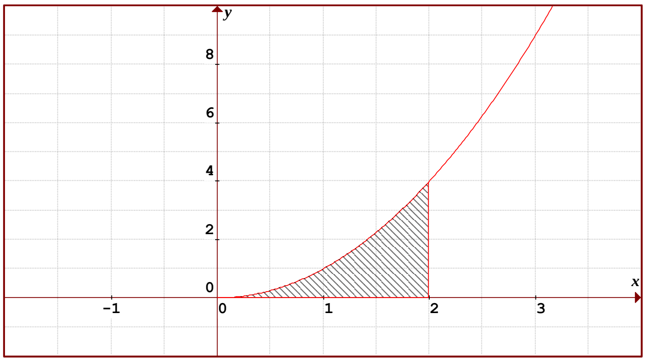
The area enclosed between a curve, the X-axis and any two values of $x$ (represented as $a$ and $b$) may be found by using definite integrals.

$$\text{Area} = \int_a^b f(x)\,dx$$

**Where:** $a$ represents the lower limit or the lower value of $x$.
$b$ represents the upper limit or the higher value of $x$.

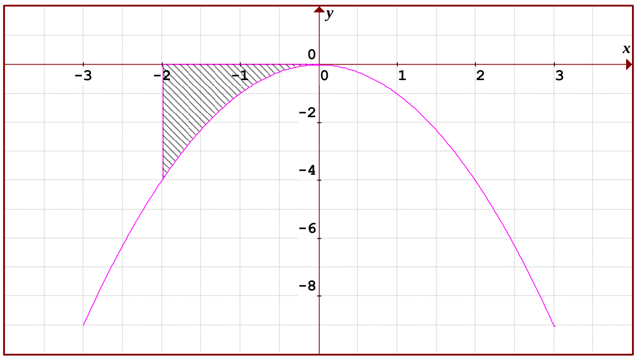
**Signs:**

The area between $x=0$ and $x=2$ in the diagram below is given by $\int_0^2 \left(x^2\right) dx$. This integral will be signed as a positive value as the corresponding region is located above the X axis.



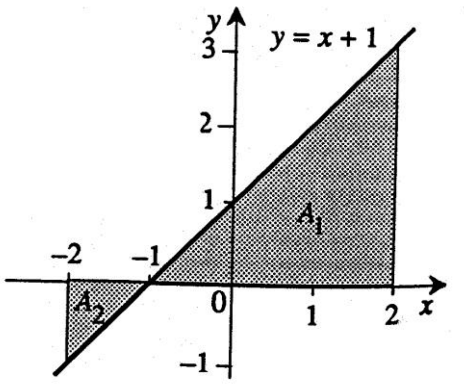

The area between $x=0$ and $x=-2$ in the diagram below is given by $\left|\int_{-2}^0 \left(-x^2\right) dx\right|$.

The absolute value of the integral $\int_{-2}^0 \left(-x^2\right) dx$ is required as a negative value will be obtained due to the fact that the corresponding region is located below the X axis (areas must be positive in value).



If part of an area falls above and part falls below the X axis, the integration process will take the sum of the signed areas into consideration. This result is referred to as the **signed area**, and does not represent the **true area** enclosed by a graph and the X axis.

As an example, consider the graph below.




Region $A_1$ carries an area of $4.5\,units^2$, whereas Region $A_2$ has an area of $0.5\,units^2$.
The total area enclosed by the line $y = x + 1$, the X axis and the lines $x = -2$ and $x = 2$ is therefore $5\,units^2$.

$$A_1 = \frac{1}{2} \times b \times h = \frac{1}{2} \times 3 \times 3 = 4.5$$

$$A_2 = \frac{1}{2} \times b \times h = \left| \frac{1}{2} \times -1 \times 1 \right| = |-0.5|$$

Integrating between $x = -2$ and $x = 2$ results in an answer of 4 square units, which clearly does not represent the true area shaded in the diagram above.

$$\int_{-2}^{2} (x+1)\, dx = \left[ \frac{x^2}{2} + x \right]_{-2}^{2} = (2+2) - (2-2) = 4$$

To find the true area of a region that lies partly above and partly below the X axis, it is necessary to integrate each region that falls above and below the X axis individually, and then add the corresponding areas without consideration to the signs.

Region 1: $\int_{-1}^{2} (x+1)\, dx = \left[ \frac{x^2}{2} + x \right]_{-1}^{2} = 4.5$

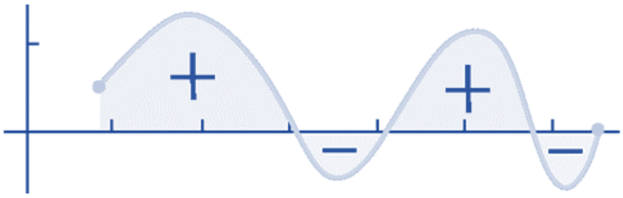
Region 2: $\int_{-2}^{-1} (x+1)\, dx = \left[ \frac{x^2}{2} + x \right]_{-2}^{-1} = -0.5$

Area = $\left| \int_{-1}^{2} (x+1)\, dx \right| + \left| \int_{-2}^{-1} (x+1)\, dx \right| = 4.5 + 0.5$ square units.

## SUMMARY

Given Area $= \int_a^b f(x)\,dx$ i.e. integrating from $a$ to $b$

Regions that fall above the $X$ axis are signed positive.



Regions that fall below the $X$ axis are signed negative.

- True areas, or simply **areas**, are always positive values.
- When we take the sign of an area into consideration, the result is called a **signed area**.
- When the region of interest is entirely **above** the X axis:

  *The True Area = The Signed Area*

- When the region of interest is entirely **below** the X axis:

  *The True Area* $\neq$ *The Signed Area*

## CALCULATING UNSIGNED AREAS OR AREAS

To calculate the area of a region, it is necessary to integrate each of the regions that fall above or below the X axis individually.

### METHOD:

**Step 1:** Sketch the curve labelling all X-intercepts. Shade in the required region.

**Step 2:** Write separate integrals for each region that falls above or below the X axis.

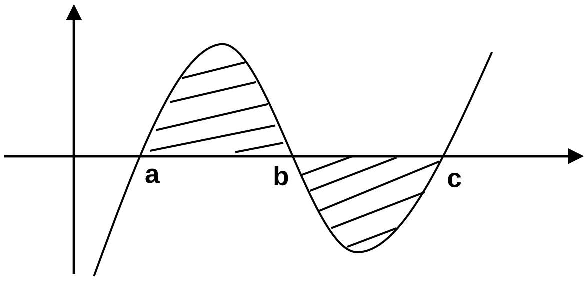
- Write integrals in a format that will result in positive values.
- Use the X-intercepts as the values of $a$ and $b$.

**Step 3:** Add the areas together.

**Note:** The calculator can only be used directly to find the **signed area.**

To obtain the true area enclosed by a graph and the X axis, we must find the definite integral for each of the regions that fall above and below the X axis.

## NOTATIONS USED TO DESCRIBE NEGATIVE AREAS




The area enclosed by the graph above and the X axis can be represented in 3 different ways.

(a) $\int_a^b [f(x)]\,dx + \left|\int_b^c [f(x)]\,dx\right|$

Positive values are obtained by taking the absolute value of negative answers.

(b) $\int_a^b [f(x)]\,dx - \int_b^c [f(x)]\,dx$

Negatively signed areas may be subtracted to produce a positive answer.

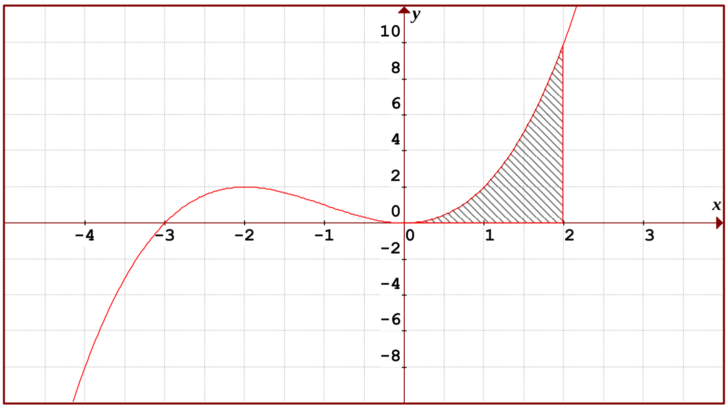
(c) $\int_a^b [f(x)]\,dx + \int_c^b [f(x)]\,dx$

The sign of an integral changes when the upper and lower limits are interchanged.

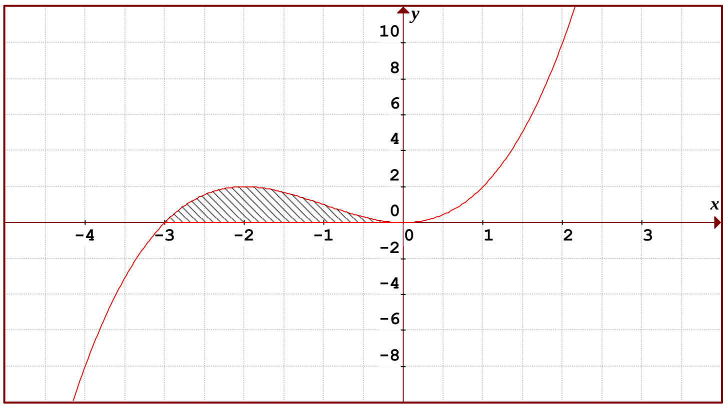
**QUESTION 47 – EXAM 1**

Given that $y = f(x)$, write integrals to represent the areas of the regions shaded below.

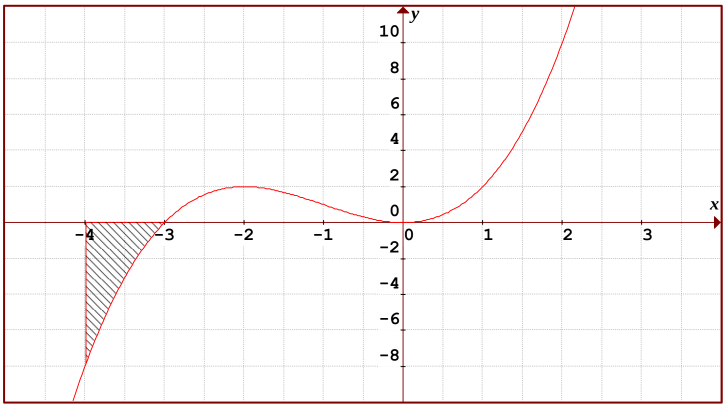
(a)



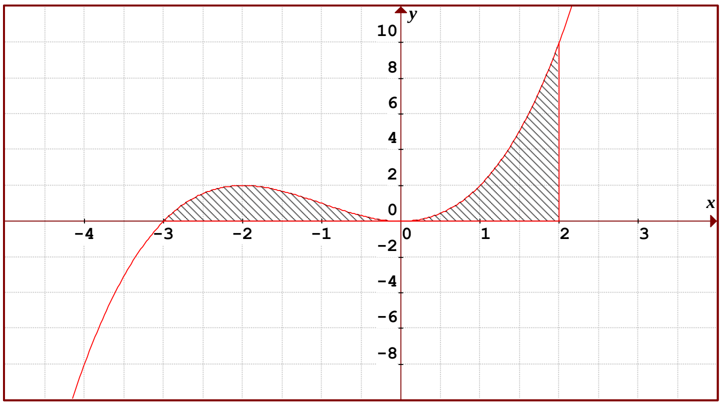

(b)




(c)

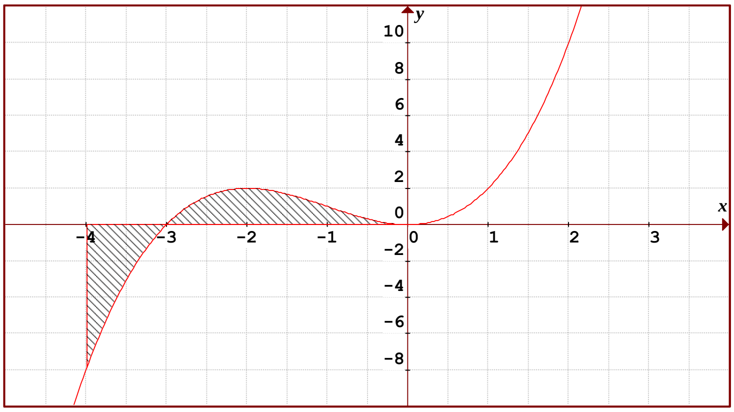


(d)



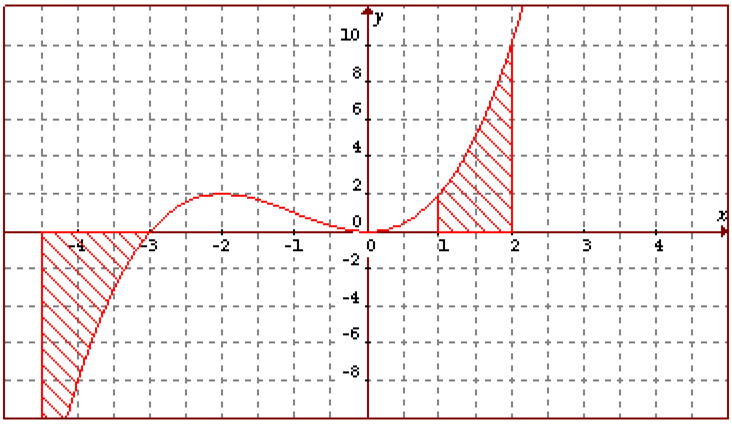
y
x
10
8
6
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3


(e)



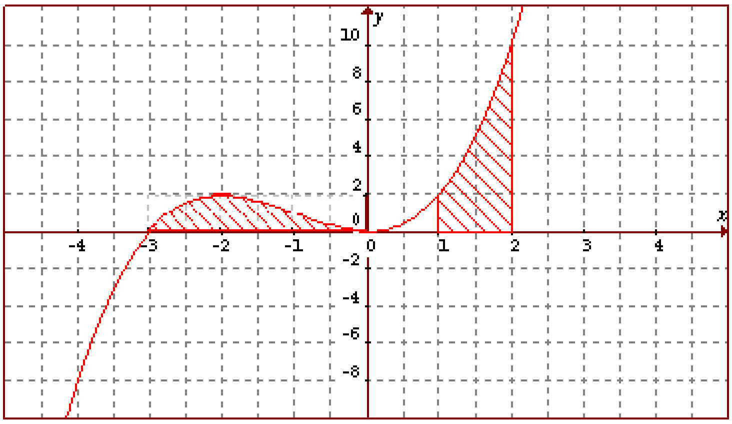
y
x
10
8
6
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3


(f)



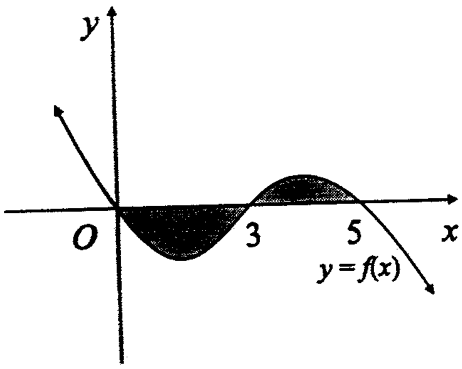
y
x
10
8
6
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3
4

(g)




**QUESTION 48 – EXAM 2**

The signed area of the shaded region is given by:




A $\int_{0}^{5} f(x)\,dx$

B $\int_{3}^{0} f(x)\,dx + \int_{3}^{5} f(x)\,dx$

C $\int_{0}^{3} f(x)\,dx - \int_{3}^{5} f(x)\,dx$

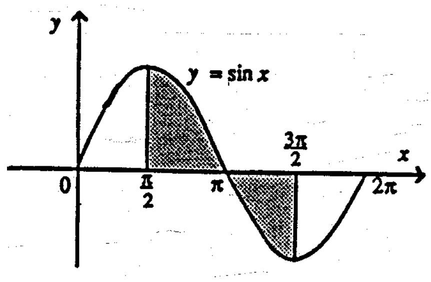
D $\int_{3}^{5} f(x)\,dx - \int_{0}^{3} f(x)\,dx$

E $-\int_{0}^{3} f(x)\,dx - \int_{3}^{5} f(x)\,dx$

***Solution***

**QUESTION 49 – EXAM 2**

The area of the shaded region of the graph is given by:




A $\displaystyle\int_{\pi/2}^{3\pi/2} \sin x \, dx$

B $\displaystyle\int_{\pi}^{3\pi/2} \sin x \, dx + \int_{\pi/2}^{\pi} \sin x \, dx$

C $\displaystyle\int_{3\pi/2}^{\pi} \sin x \, dx + \int_{\pi/2}^{\pi} \sin x \, dx$

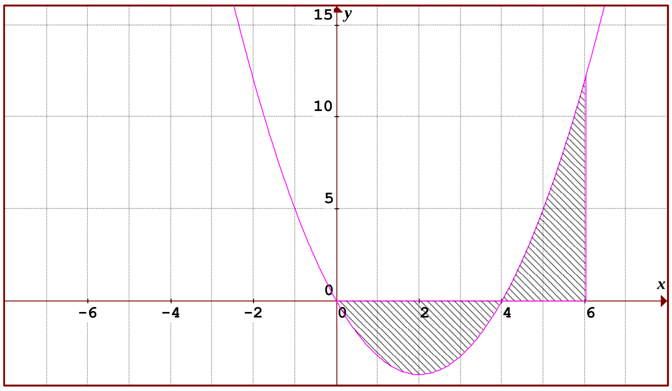
D $\displaystyle\int_{\pi/2}^{3\pi/2} \sin x \, dx + \int_{\pi}^{\pi/2} \sin x \, dx$

E $\displaystyle\pi \int_{\pi/2}^{3\pi/2} \sin^2 x \, dx$

***Solution***

## QUESTION 50 – EXAM 1

Using calculus, find the area of the region enclosed by the graph of $y = x^2 - 4x$, the X axis and the lines $x = 0$ and $x = 6$.




***Solution***

**Step 1:** Sketch the curve labelling all X-intercepts. Shade in the required region.

**Step 2:** Write separate integrals for each region that falls above or below the X axis.

- Write integrals in a format that will result in postive values.
- Use the X-intercepts as the values of $a$ and $b$.

Area = $\left|\int_0^4 (x^2 - 4x)\,dx\right| + \int_4^6 (x^2 - 4x)\,dx$

$$= \left|\left[\frac{x^3}{3} - 2x^2\right]_0^4\right| + \left[\frac{x^3}{3} - 2x^2\right]_4^6$$

$$= \left|\left\{\left(\frac{64}{3} - 32\right) - (0)\right\}\right| + \left\{\left(\frac{216}{3} - 72\right) - \left(\frac{64}{3} - 32\right)\right\}$$

**Step 3:** Add the areas together.

Area $= \left|-\frac{32}{3}\right| + \frac{32}{3} = \frac{64}{3}\ units^2$

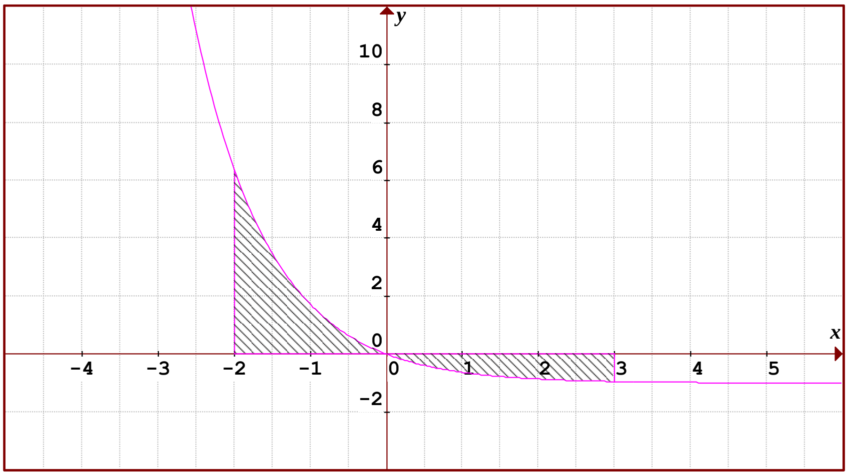
**Note:** **Alternate notations for the separate integrals include**

Area = $\int_4^0 (x^2 - 4x)\,dx + \int_4^6 (x^2 - 4x)\,dx$

Area = $\int_4^6 (x^2 - 4x)\,dx - \int_0^4 (x^2 - 4x)\,dx$

**QUESTION 51 – EXAM 1 & 2**

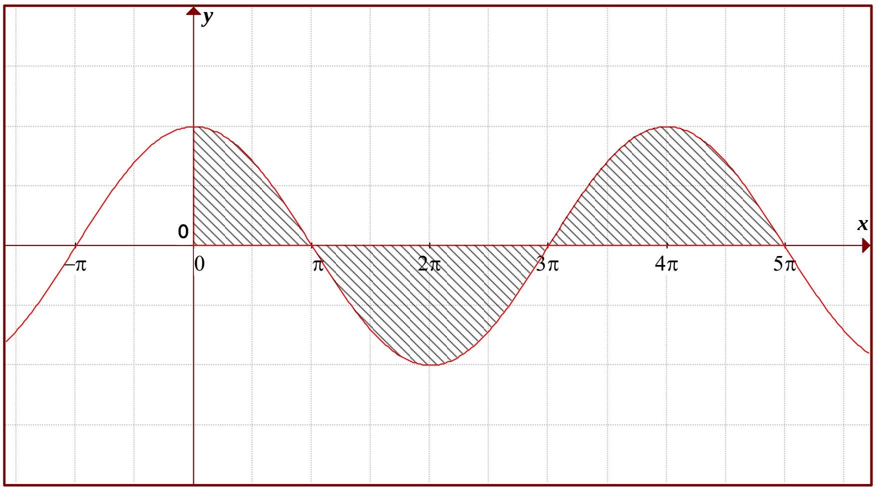
Show that the area enclosed by graph of $y = e^{-x} - 1$ and the lines $x = -2$ and $x = 3$ is $e^{2} + e^{-3} - 1\ units^{2}$.




***Solution***

## QUESTION 52 – EXAM 1

Use calculus to find the area enclosed by graph of $y = \cos\left(\frac{x}{2}\right)$ and the lines $x = 0$ and $x = 5\pi$.




***Solution***

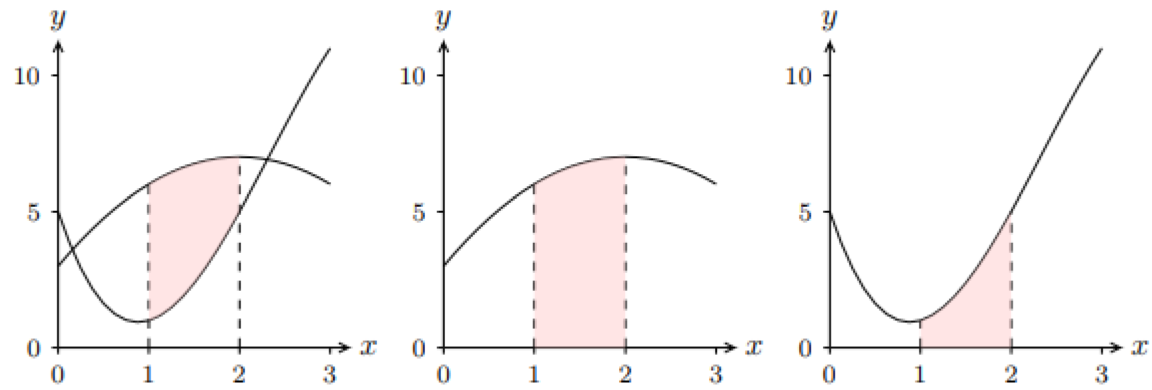
## QUESTION 53 – EXAM 1

Using calculus, find the area enclosed between the curve $f(x)=\frac{1}{x-2}$, the X axis, and the lines $x=1$ and $x=-1$.

***Solution***

## AREAS BETWEEN CURVES

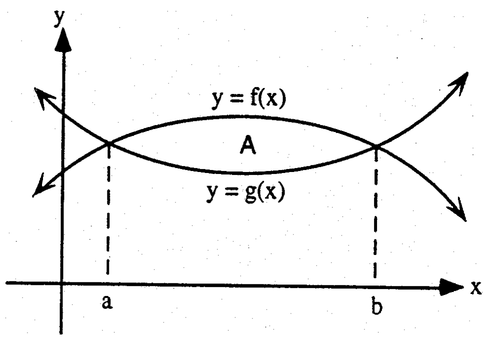
The area between two curves denoted as $f(x)$ and $g(x)$ is the difference in the areas enclosed between each individual curve and the X axis.




Area = $\int_a^b$ ( the equation of the upper curve ) – ( the equation of the lower curve ) $dx$

$$A = \int_a^b [f(x) - g(x)]dx$$

**Note:** $a$ and $b$ generally represent the points of intersection of the two curves.




**Note:**

There is no need to split integrals if the required area falls above and below the X axis. Only when the equation of the upper or lower curve changes is it necessary to write separate integrals for each of the different regions.

## METHOD:

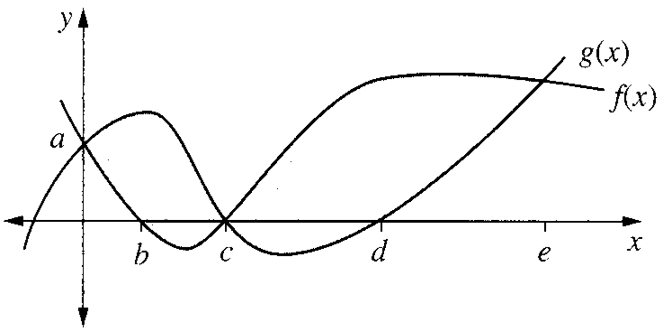
**Step 1:** Sketch the curves and shade in the appropriate region.

**Step 2:** If the upper and lower limits are not given, find the points of intersection of the two curves.

**Step 3:** Ensure that the equation of the upper curve does not change over the required region. If the equation of the upper curve changes write separate integrals for each individual section of the curve.

**Step 4:** Evaluate each integral and add the absolute values of each area.

**QUESTION 54 – EXAM 2**




The area enclosed by the curves $f(x)$ and $g(x)$ and the lines $x=0$ and $x=e$ is best represented by:

**A** $\int_0^c g(x)-f(x)dx+\int_c^d f(x)-g(x)dx$

**B** $\int_0^c g(x)-f(x)dx-\int_e^c f(x)-g(x)dx$

**C** $\int_0^c g(x)dx+\int_b^c f(x)dx+\int_c^d g(x)dx+\int_d^e f(x)dx$

**D** $\int_0^e f(x)dx+\int_0^e g(x)dx$

**E** $\int_0^a f(x)-g(x)dx+\int_c^e g(x)-f(x)dx$

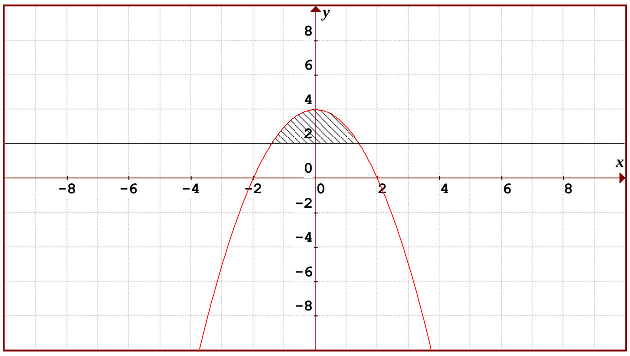
***Solution***

## QUESTION 55 – EXAM 1

Find the area enclosed by the graphs of $f(x) = 2$ and $f(x) = 4 - x^2$.

***Solution***

**Step 1:** Sketch the curves and shade in the appropriate region.




**Step 2:** If the upper and lower limits are not given, find the points of intersection of the two curves.

Let $4 - x^2 = 2$

$\therefore -x^2 = -2$

$x^2 = 2$

$\therefore x = \pm\sqrt{2}$

**Step 3:** Evaluate the area(s). Ensure that the equation of the upper curve does not change over the required region. If the equation of the upper curve changes, write separate integrals for each individual section of the curve.

Area = $\int_{-\sqrt{2}}^{\sqrt{2}} [(4 - x^2) - 2]\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)\,dx = \left[2x - \frac{x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}$

$$= \left[2\sqrt{2} - \frac{(\sqrt{2})^3}{3}\right] - \left[-2\sqrt{2} - \frac{(-\sqrt{2})^3}{3}\right]$$

$$= \left[2\sqrt{2} - \frac{2\sqrt{2}}{3}\right] - \left[-2\sqrt{2} + \frac{2\sqrt{2}}{3}\right] = 2\sqrt{2} - \frac{2\sqrt{2}}{3} + 2\sqrt{2} - \frac{2\sqrt{2}}{3}$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{3} = \frac{8\sqrt{2}}{3} \text{ square units}$$

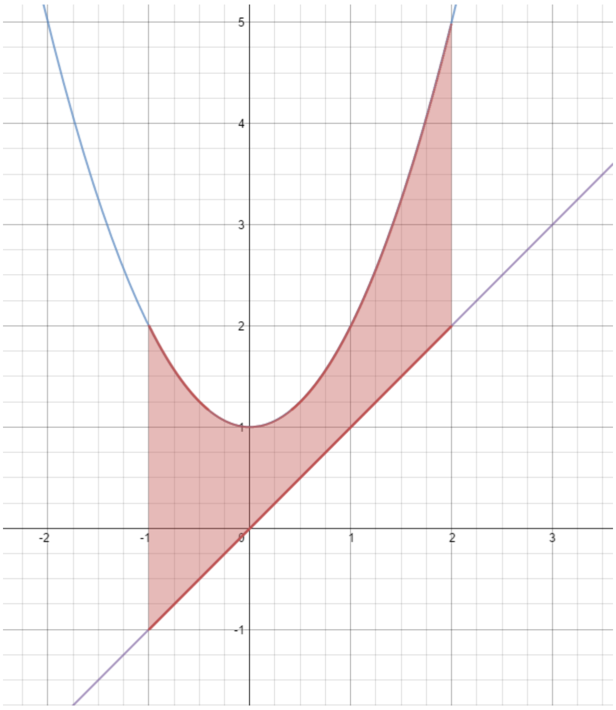
**Alternatively, use symmetry properties to simplify the solution process:**

Area = $\int_{-\sqrt{2}}^{\sqrt{2}} [(4 - x^2) - 2]\,dx = 2\int_{0}^{\sqrt{2}} (2 - x^2)\,dx = 2\left[2x - \frac{x^3}{3}\right]_{0}^{\sqrt{2}} = 2\left[2\sqrt{2} - \frac{(\sqrt{2})^3}{3}\right] = \frac{8\sqrt{2}}{3}$ units$^2$

**QUESTION 56 – EXAM 1**

Find the area enclosed by the graphs of $f(x) = x$ and $f(x) = x^2 + 1$ and the lines $x = -1$ and $x = 2$.

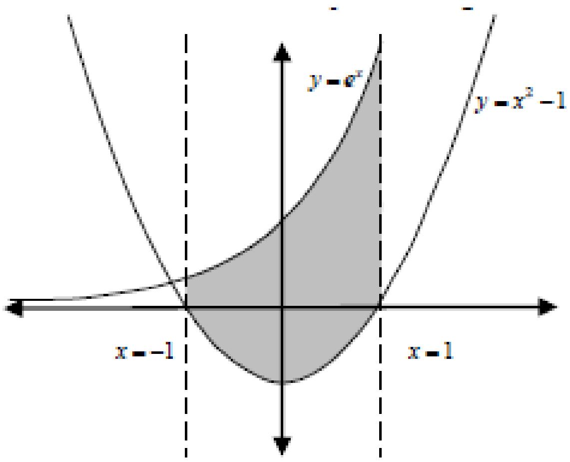
***Solution***



**QUESTION 57 – EXAM 1**

Find the area of the region enclosed by the following curves: $y_1 = e^x$, $y_2 = x^2 - 1$, $x = -1$ and $x = 1$.

***Solution***

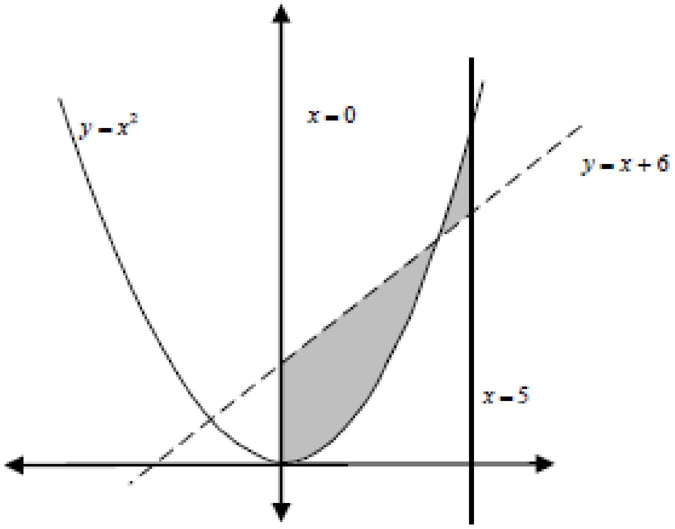


**QUESTION 58 – EXAM 1**

Find the area enclosed by the graphs of $f(x) = x^2$, $f(x) = x + 6$, $x = 0$ and $x = 5$.

***Solution***

**Step 1:** Sketch the curves and shade in the appropriate region.




**Step 2:** If the upper and lower limits are not given, find the points of intersection of the two curves.

Points of intersection: $x^2 = x + 6$

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = 3, -2$$

**Step 3:** Evaluate the area(s). Ensure that the equation of the upper curve does not change over the required region. If the equation of the upper curve changes, write separate integrals for each individual section of the curve.

$$A = \int_0^3 \left(x + 6 - x^2\right) dx + \int_3^5 \left(x^2 - (x+6)\right) dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3} - \frac{x^2}{2} - 6x\right]_3^5$$

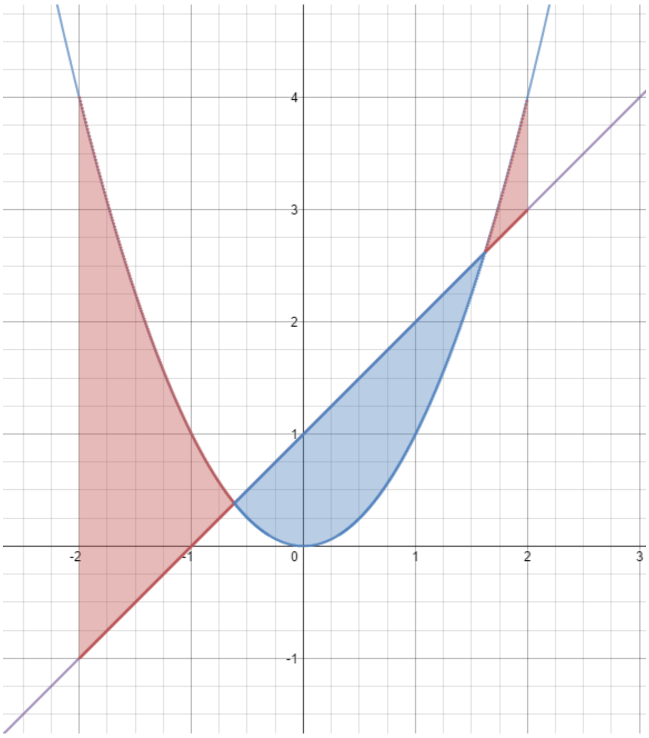
$$= \left(\frac{9}{2} + 18 - 9\right) + \left(\frac{125}{3} - \frac{25}{2} - 30\right) - \left(9 - \frac{9}{2} - 18\right)$$

$$= \frac{157}{6} units^2$$

## QUESTION 59 – EXAM 2

Using calculus, find the area enclosed by the graphs of $f(x)=x^2$ and $f(x)=x+1$ and the lines $x=-2$ and $x=2$. State your answer correct to 3 decimal places.

***Solution***



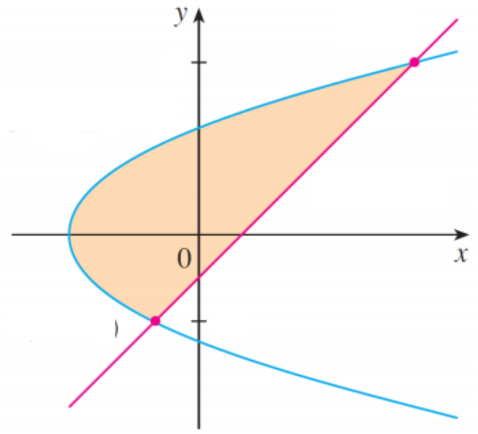
**QUESTION 60 – EXAM 2**

Find the area enclosed by the graphs $y = e^{2x}$, $y = e^{-3x}$ and the lines $x = -2$ and $x = 3$.

***Solution***

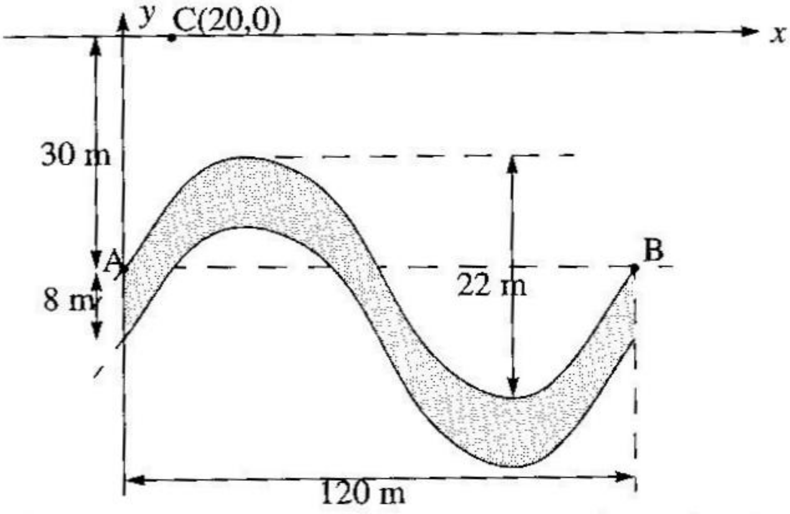
**QUESTION 61 – EXAM 2**

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.




***Solution***

**QUESTION 62 – EXAM 2**
A layer of ore beneath the ground in outback Australia has surfaces that are sinusoidal in cross section. A miner has drawn up a rough sketch on a set of axes that has been set up so that the X axis represents the ground surface.




The miner feels that an appropriate equation to represent the top level of the sinusoidal curve, relative to the set of axes she has drawn is

$$y = a\sin(kx) + d, \quad 0 \le x \le 120.$$

(a) Show that $a = 11$, $d = -30$ and $k = \dfrac{\pi}{60}$. (3 marks)

(b) What is the minimum distance that miners will need to drill to reach the ore? (1 mark)

(c) How deep must miners drill when they are at Point C, if they must drill through the layer? (2 marks)

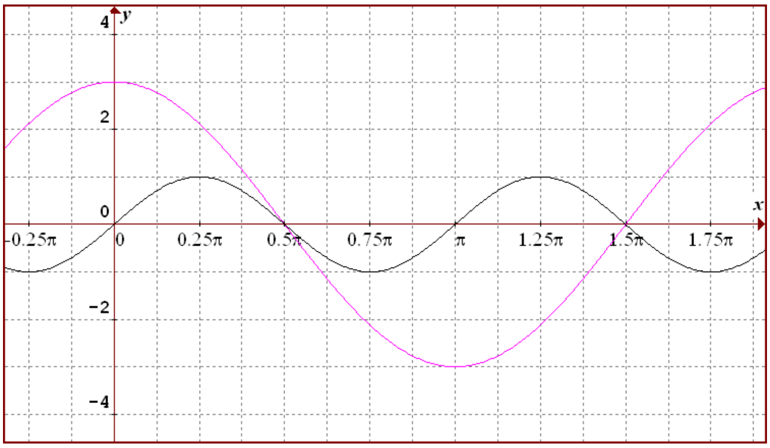
Once the miners have drilled at Point C and have reached the lower layer, a second pipe is to run horizontally until it hits the layer of ore again.

(d) What is the minimum length that this horizontal pipe will need to be? (2 marks)

(e) Use calculus to find the exact cross sectional area of the dirt that lies between ground level and the top layer of the ore for $0 \leq x(m) \leq 120$. (3 marks)

(f) Hence find the volume of dirt that would need to be removed to expose the layer of ore if the width of the mining site was 15 metres. (1 mark)

## QUESTION 63 – EXAM 2




The area of the region enclosed by the graphs of $f(x) = 3\cos(x)$ and $g(x) = \sin(2x)$ and the lines $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ can be written as $\int_a^b (c - d)\, dx$ where $a, b$ represent real number constants and $c, d$ represent functions in terms of $x$.

(a) Find $a, b, c$ and $d$ and hence calculate the area enclosed by the graphs.

$a = \frac{\pi}{2}$, $b = \frac{3\pi}{2}$, $c = \sin 2x$, $d = 3\cos x$

(b) The area of the region enclosed by the graphs of $f(x)=3\cos(x)$ and $g(x)=\sin(2x)$ and the lines $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ can also be written as $e\left|\int_f^g h\cos(x)\,dx\right|$ where $e, f, g, h$ represent real number constants. State the values of $e, f, g, h$.

**QUESTION 64 – EXAM 1**

Find the value(s) of $b$ such that the area of the region enclosed by the parabolas $y = x^2 - b^2$ and $y = b^2 - x^2$ is 72.

***Solution***