NEWTON'S LAW OF COOLING

Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference between its temperature and that of its immediate surroundings. Therefore:

$$\frac{dT}{dt} \propto -(T - T_0)$$

This always results in a first order differential equation of Type 2:

$$\boxed{\frac{dT}{dt} = -k(T - T_0)}$$

Where:
- T is the body's temperature at time t.
- T_0 is the temperature of the surroundings.

The general solution is: $\boxed{T = Ae^{-kt} + T_0}$

When $t = 0$ then $T = A + T_0$. Therefore, in this particular model, the initial value of T is given by $T = A + T_0$.

QUESTION 37

A body originally at a temperature of 90°C is placed into a room with a temperature of 25°C. The body cools down to 70°C in 9 minutes.

(a) Find the **exact** temperature of the body after 15 minutes.

Let T represent the temperature in degrees Celsius of the body after t minutes.

Solve the DE:

$\frac{dT}{dt} = -k(T - 25)$, subject to the conditions $T(0) = 90$ and $T(9) = 70$,

$$\therefore \frac{dt}{dT} = -\frac{1}{k} \times \frac{1}{(T - 25)}$$

$$\therefore t = -\frac{1}{k}\int \frac{1}{(T - 25)}\, dT = -\frac{1}{k}\log_e |T - 25| + C$$

$\therefore T = e^{k(C-t)} + 25 = e^{kC}e^{-kt} + 25 = Ae^{-kt} + 25$ where $A = e^{kC}$

To find the values of the constants A and k, substitute the given conditions T = 90° C when t = 0 and T = 70° C when t = 9 minutes.

T = 90°C when t = 0: $65 = A$

Update the general solution by substituting A = 65:

$$T = 65e^{-kt} + 25$$

T = 70° C when t = 9: $45 = 65e^{-9k}$

$$\therefore \frac{9}{13} = e^{-9k}$$

$$\therefore k = -\frac{1}{9}\log_e \frac{9}{13} = \frac{1}{9}\log_e \frac{13}{9}$$

Therefore $T = 65e^{-kt} + 25$, where $k = \frac{1}{9}\log_e \frac{13}{9}$.

Note: Life is easier if you **do not** substitute the value for k directly into the expression for T.

Determine the temperature of the body after 15 minutes.

Substitute t = 15 into $T = 65e^{-kt} + 25$, where $k = \frac{1}{9}\log_e \frac{13}{9}$, and solve for T:

$$T = 65e^{-\frac{15}{9}\log_e \frac{13}{9}} + 25$$

$$= 65e^{\log_e \left(\frac{13}{9}\right)^{-\frac{5}{3}}} + 25$$

$$= 65\left(\frac{13}{9}\right)^{-\frac{5}{3}} + 25$$

$\therefore \quad T = 65\left(\frac{9}{13}\right)^{\frac{5}{3}} + 25\ °C$ ($\approx 60^0 C$, correct to the nearest degree.)

Note: If calculating an answer to a specified degree of accuracy, the exact value of k should be used during the calculation in order to avoid accumulation of rounding error.

(b) Determine **exactly** how long it takes for the body to cool to a temperature of 48°C.

To determine how long it takes for the body to cool to a temperature of 48°C, substitute T = 48 into $T = 65e^{-kt} + 25$, where $k = \frac{1}{9}\log_e \frac{13}{9}$, and solve for t:

$$23 = 65e^{-kt}$$

$$\therefore \frac{23}{65} = e^{-kt}$$

$$\therefore t = \frac{-\log_e \frac{23}{65}}{k} = \frac{\log_e \frac{65}{23}}{k}$$ minutes, where $k = \frac{1}{9}\log_e \frac{13}{9}$

$$t = \frac{\log_e\left(\frac{65}{23}\right)}{\frac{1}{9}\log_e\left(\frac{13}{9}\right)}$$

$$\therefore t = \frac{9\log_e\left(\frac{65}{23}\right)}{\log_e\left(\frac{13}{9}\right)}$$ mins

(≈ 25 minutes and 26 seconds, correct to the nearest second.)

QUESTION 38

Mr Cal Culus makes himself a cup of coffee. He likes his coffee hot and will not drink it if it has a temperature lower than 55°C. If the coffee initially had a temperature of 83°C and the room's temperature is 19°C, how long, to the nearest second, does he have before the coffee is too cold given that the temperature of the coffee was 70°C after 3 minutes. (Assume Newton's Law of Cooling).

Solution

Many situations involve a difference of rates. In such cases:

Rate of change = (rate of **increase**) – (rate of **decrease**).

INPUT/OUTPUT OF MIXING PROBLEMS

The term **mixing problem** is used to describe a problem that involves changes in the concentration of the components of a mixture.

The rate of increase/decrease of the amount of a substance in a solution is equal to the difference between the rate of inflow (or input) and the rate of outflow (or output) of the substance. If x represents the amount of substance in a solution at any time t then:

$$\frac{dx}{dt} = IR - OR$$

Where:

IR = Rate of inflow of substance

= (inflow volume rate) $\times$ (inflow concentration)

OR = Rate of outflow of substance

= (outflow volume rate) $\times$ (outflow concentration)

The concentration of a substance, C, is the mass of substance (m) divided by the volume of mixture (V); that is: $C = \frac{m}{V}$.

QUESTION 39

A tank initially contains 15 kg of salt dissolved in 100 litres of water. Into this tank is flowing, at a rate of 4 litres per minute, a salt solution containing 2 kg of salt per litre. Assuming that the mixture is kept uniform and that after t minutes, m kg of salt remains dissolved in the tank:

(i) Construct an appropriate differential equation expressing $\frac{dm}{dt}$ as a function of t.

(ii) Find an expression of m in terms of t.

If the mixture:

(a) Remains in the tank.

(b) Flows out at the rate of 4 litres per minute.

QUESTION 40

A tank initially contains 400 litres of water in which is dissolved, 10 kg of sugar. A sugar solution of concentration 0.2 kg/L is poured into the tank at the rate of 2 L/min. The mixture, which is kept uniform by stirring, flows out at the rate of 2 L/min. If the mass of sugar in the tank after t minutes is $Q\ kg$:

(a) Find, correct to 2 decimal places, the amount of sugar in the tank after 7 minutes.

(b) How long, correct to the nearest second, does it take for the amount of sugar to be doubled?

QUESTION 41

A tank initially contains 16 kg of salt dissolved in 90 litres of water. A salt solution containing 6 kg of salt per litre is flowing into this tank at a rate of 5 litres per minute. The mixture is kept uniform by stirring.

(a) Set up (but do not solve) the differential equation if the mixture flows out at 7 litres per minute.

Let m represent the amount of salt in the tank after t minutes.

Step 1: Write down a general expression for the rate of change of amount of salt in the tank after t minutes:

$\frac{dm}{dt}$ = (amount of salt entering tank each minute)

$-$ (amount of salt leaving tank each minute)

Step 2: Determine an expression for the quantity '*amount of salt entering tank each minute*':

By virtue of the inflow:

Amount of salt entering tank each minute

= (rate of inflow) $\times$ (concentration of salt in inflow)
= (5 litres per minute) $\times$ (6 kilograms per litre)
= 30 kilograms per minute

Step 3: Determine an expression for the quantity '*amount of salt leaving tank each minute*':

By virtue of the outflow:

Amount of salt leaving tank each minute

= (rate of outflow) $\times$ (concentration of salt in outflow)
= (7 litres per minute) $\times$ (concentration of salt in outflow)

Step 4: Determine an expression for the quantity '*concentration of salt in outflow*':

After t minutes have elapsed: Concentration of salt in outflow = $\frac{m}{V}$,

where V is the volume of solution in the tank after t minutes. But:

V = (initial volume)
+ (volume added after t minutes via inflow)
– (volume removed after t minutes via outflow)

= 90 litres + 5t litres – 7t litres

= (90 – 2t) litres

Therefore:

Concentration of salt in outflow = $\frac{m}{V} = \frac{m}{90-2t}$ grams per litre.

Substitute this expression into Step 3:

Amount of salt leaving tank each minute

= (7 litres per minute) × (concentration of salt in outflow)

= (7 litres per minute) × $\left(\frac{m}{90-2t} \text{ kilograms per litre}\right)$

= $\left(\frac{7m}{90-2t} \text{ kilograms per minute}\right)$

Step 5: Substitute the expressions from Step 2 and Step 4 into Step 1:

$\frac{dm}{dt}$ = (amount of salt entering tank each minute)
– (amount of salt leaving tank each minute)

$$= (30 \text{ kilograms per minute}) - \left(\frac{7m}{90-2t} \text{ kilograms per minute}\right)$$

$\frac{dm}{dt} = 30 - \frac{7m}{90-2t}$, subject to the condition $m(0) = 16$

(b) Set up (but do not solve) the differential equation if the mixture flows out at 3 litres per minute.

Step 1: Write down a general expression for the rate of change of amount of salt in the tank after t minutes:

$\frac{dm}{dt}$ = (amount of salt entering tank each minute)

– (amount of salt leaving tank each minute)

Step 2: Determine an expression for the quantity '*amount of salt entering tank each minute*':

By virtue of the inflow:

Amount of salt entering tank each minute

= (rate of inflow) $\times$ (concentration of salt in inflow)
= (5 litres per minute) $\times$ (6 kilograms per litre)
= 30 kilograms per minute

Step 3: Determine an expression for the quantity '*amount of salt leaving tank each minut*e':

By virtue of the outflow:

Amount of salt leaving tank each minute

= (rate of outflow) $\times$ (concentration of salt in outflow)
= (3 litres per minute) $\times$ (concentration of salt in outflow)

Step 4: Determine an expression for the quantity '*concentration of salt in outflow*':

After t minutes have elapsed:

Concentration of salt in outflow = $\frac{m}{V}$,

where V is the volume of solution in the tank after t minutes. But:

V = (initial volume)
+ (volume added after t minutes via inflow)
– (volume removed after t minutes via outflow)

= 90 litres + $5t$ litres – $3t$ litres

= $(90 + 2t)$ litres

Therefore: Concentration of salt in outflow = $\frac{m}{V} = \frac{m}{90+2t}$ grams per litre.

Substitute this expression into Step 3:

Amount of salt leaving tank each minute

= (3 litres per minute) × (concentration of salt in outflow)

= (3 litres per minute) × $\left(\frac{m}{90+2t} \text{ kilograms per litre}\right)$

= $\left(\frac{3m}{90+2t} \text{ kilograms per minute}\right)$

Step 5: Substitute the expressions from Step 2 and Step 4 into Step 1:

$\frac{dm}{dt}$ = (amount of salt entering tank each minute)

− (amount of salt leaving tank each minute)

= $(30 \text{ kilograms per minute}) - \left(\frac{3m}{90+2t} \text{ kilograms per minute}\right)$

$\frac{dm}{dt} = 30 - \frac{3m}{90+2t}$, subject to the condition $m(0) = 16$

(c) Set up and solve the differential equation for the quantity of salt in the tank at time t if the mixture flows out at 5 litres per minute.

Let m represent the amount of salt in the tank after t minutes.

Step 1: Write down a general expression for the rate of change of amount of salt in the tank after t minutes:

$\frac{dm}{dt}$ = (amount of salt entering tank each minute)

– (amount of salt leaving tank each minute)

Step 2: Determine an expression for the quantity '*amount of salt entering tank each minute*':

By virtue of the inflow:

Amount of salt entering tank each minute

= (rate of inflow) × (concentration of salt in inflow)
= (5 litres per minute) × (6 kilograms per litre)
= 30 kilograms per minute

Step 3: Determine an expression for the quantity '*amount of salt leaving tank each minute*':

By virtue of the outflow:

Amount of salt leaving tank each minute

= (rate of outflow) × (concentration of salt in outflow)
= (5 litres per minute) × (concentration of salt in outflow)

Step 4: Determine an expression for the quantity '*concentration of salt in outflow*':

After t minutes have elapsed:

Concentration of salt in outflow = $\frac{m}{V}$, where V is the volume of solution in the tank after t minutes. But:

V = (initial volume)
+ (volume added after t minutes via inflow)
– (volume removed after t minutes via outflow)

= 90 litres + 5t litres – 5t litres = 90 litres

Therefore:

Concentration of salt in outflow = $\frac{m}{V} = \frac{m}{90}$ grams per litre.

Substitute this expression into Step 3:

Amount of salt leaving tank each minute:

= (5 litres per minute) $\times$ (concentration of salt in outflow)

= (5 litres per minute) $\times \left(\frac{m}{90} \text{ kilograms per litre} \right)$

= $\left(\frac{m}{18} \text{ kilograms per minute} \right)$

Step 5: Substitute the expressions from Step 2 and Step 4 into Step 1:

$\frac{dm}{dt}$ = (amount of salt entering tank each minute)

$-$ (amount of salt leaving tank each minute)

$= (30 \text{ kilograms per minute}) - \left(\frac{m}{18} \text{ kilograms per minute} \right)$

$\frac{dm}{dt} = 30 - \frac{m}{18} = \frac{540 - m}{18}$, subject to the condition $m(0) = 16$.

Step 6: Solve $\frac{dm}{dt} = \frac{540 - m}{18}$, subject to the condition $m(0) = 16$:

Take the reciprocal of both sides: $\frac{dt}{dm} = \frac{18}{540 - m}$.

Anti-differentiate with respect to the variable m:

$$t = 18 \int \frac{1}{540 - m} \, dm = -18 \log_e |540 - m| + C$$

Re-arrange to make m the subject:

$$m = 540 - e^{(C-t)/18} = 540 - e^{C/18} e^{-t/18} = 540 - Ae^{-t/18}$$

Substitute the initial condition $m = 16$ when $t = 0$ and solve for *A*: $524 = A$

$$\therefore m = 540 - 524e^{-t/18}$$

QUESTION 42

A rectangular vessel is divided into two equal compartments by a vertical porous membrane. Liquid in one compartment, initially at a depth of 20 cm, passes into the other compartment, initially empty, at a rate proportional to the difference in levels.

(a) If the depth of liquid in one vessel at any time, t minutes, is x cm, find an expression for the rate of change in depth as a function to time between the two compartments.

(b) Hence show that $x = 10(1 - e^{-2kt})$.

(c) If the level in the second compartment rises 2 cm in the first 5 minutes, after what time, to the nearest second, will the difference in levels be 2 cm?